Lesson 24 Power Series

Last class: geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

← This form is helpful for students

$$= \frac{a}{1-r} \quad \text{if } |r| < 1$$

$\frac{\text{first term}}{1-\text{ratio}}$

diverges if $|r| \geq 1$

Our series in the last class only contained #'s.
Today: series will contain variables

$\boxed{Ex}$ $$\sum_{n=0}^{\infty} x^n = \underbrace{1 + x + x^2 + x^3 + \cdots}_{\text{function of } x}$$

✓ Looks like an infinite polynomial

This series w/ vars is geometric: $a = 1$ $r = x$
(Not all are geometric)

(D) A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

c_i - constants
x - variable

- Look like and act like infinite polynomials
- Are functions of x

closed form of the series

Ex $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \boxed{\dfrac{1}{1-x}}$

power series
$c_n = 1$ for all n

provided that

$|x| < 1$

radius of convergence

(D) The power series $\sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence R if the series converges when $|x| < R$ and diverges when $|x| > R$

Ex Find the power series rep. for

$$f(x) = \frac{2}{1-9x^2} = \frac{a}{1-r}$$

$a = 2$
$r = 9x^2$

and the radius of convergence

$$f(x) = \sum_{n=0}^{\infty} (2)(9x^2)^n = \sum_{n=0}^{\infty} 2 \cdot 9^n \cdot x^{2n}$$

ar^n

$c_n = 2 \cdot 9^n$

Helpful later to have $\sum$ constant $\cdot\, x^{\text{power}}$

Radius of Convergence: $|9x^2| < 1$
$9|x|^2 < 1$
$|x|^2 < \frac{1}{9}$

$|x| < \frac{1}{3}$

$R = \frac{1}{3}$

$\boxed{Ex}$ $g(x) = \frac{5x}{2+x^3}$

① Write as a power series
② Find radius of convergence
③ Evaluate $\int g(x)\,dx$ as a power series

① $\frac{5x}{2+x^3} \underset{=}{\text{want}} \frac{a}{1-r}$

$a = 5x \quad r = \frac{-x^3}{2}$

$$\frac{5x}{2\left(1+\frac{x^3}{2}\right)} = \frac{1}{2}\left[\frac{5x}{1+\frac{x^3}{2}}\right] = \frac{1}{2}\left[\frac{5x}{1-\left(-\frac{x^3}{2}\right)}\right]$$

$-x^3 = (-1)x^3$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(5x)\left(\frac{-x^3}{2}\right)^n$$

Exponent Rule

$a^b a^c = a^{b+c}$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\cdot 5 \cdot x \frac{(-1)^n x^{3n}}{2^n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 5}{2^{n+1}} x^{3n+1}}$$

② $|r| < 1$

$\left|\frac{-x^3}{2}\right| < 1$

$\frac{|x|^3}{2} < 1$

$|x|^3 < 2$

$|x| < \boxed{\sqrt[3]{2}}$ Radius of convergence

③ $$\int \frac{5x}{2+x^3}\,dx = \int\left(\sum_{n=0}^{\infty} \frac{(-1)^n 5}{2^{n+1}} x^{3n+1}\right)dx$$

$$\int 4x^7\,dx = \frac{4x^8}{8}$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n 5}{2^{n+1}} \frac{x^{3n+2}}{3n+2}$$

$\boxed{\text{Ex}}$ You try it!

Use the first 3 nonzero terms of the power series to approximate

$$\int_0^{0.4} \frac{3}{5+10x}\,dx$$

$a=3 \quad r=-2x$

$$\frac{3}{5+10x} = \frac{3}{5(1+2x)} = \frac{1}{5}\left(\frac{3}{1-(-2x)}\right) = \frac{1}{5}\sum_{n=0}^{\infty} 3(-2x)^n$$

$$= \sum_{n=0}^{\infty} \frac{3}{5}\underbrace{(-1)^n(2^n)}_{\text{OR } (-2)^n} x^n$$

$|-2x|<1$

$2|x|<1$

$|x|<\frac{1}{2}$

Radius of Conv.

$$\int \frac{3}{5+10x}\,dx = C + \sum_{n=0}^{\infty} \frac{3}{5}(-1)^n(2^n)\frac{x^{n+1}}{n+1}$$

$$\approx C + \frac{3}{5}x - \frac{6}{5}\frac{x^2}{2} + \frac{12}{5}\frac{x^3}{3}$$

$$\int_0^{0.4} \frac{3}{5+10x}\,dx \approx \frac{3}{5}(0.4) - \frac{6}{5}\frac{(0.4)^2}{2} + \frac{12}{5}\frac{(0.4)^3}{3} - (0-0+0)$$

$$\approx 0.1952$$